

# Correlation functions of just renormalizable tensorial group field theory: The melonic approximation

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## Abstract

The  $D$ -colored version of tensor models has been shown to admit a large  $N$ -limit expansion. The leading contributions result from so-called melonic graphs which are dual to the  $D$ -sphere. This is a note about the Schwinger-Dyson equations of the tensorial  $\varphi_5^4$ -model (with propagator  $1/\mathbf{p}^2$ ) and their melonic approximation. We derive the master equations for two- and four-point correlation functions and discuss their solution.

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## 1 Introduction

The construction of a consistent quantum theory of gravity is one of the biggest open problems of fundamental physics. There are several approaches to this challenging issue. Tensor models belong to the promising candidates to understand quantum gravity (QG) in dimension  $D \geq 3$  [1]-[4]. Tensor models come from group field theory, which is a second-quantization of the loop quantum gravity, spin foam and certainly from matrix models [5]. Tensorial group field theory (TGFT) is quantum field theory (QFT) over group manifolds.

It can also be viewed as a new proposal for quantum field theories based on a Feynman path integral, which generates random graphs describing simplicial pseudo manifolds.

A few years ago, Razvan Gurău [6]-[12] achieved a breakthrough for this program by discovering the generalization of t'Hooft's  $1/N$ -expansion [13]-[14]. This allows to understand statistical physics properties such as continuum limit, phase transitions and critical exponents (see [15]-[25] for more detail).

All field theories must be physically justified by renormalizability. In the case of tensor models, by modifying the propagator using radiative corrections of the form  $1/\mathbf{p}^2$  [26], this question has been solved under specific prescriptions [27]-[35]. The  $\beta$ -functions of such models are also derived. It has been shown that asymptotic freedom is the generic feature of all TGFT models [30] and [36]-[38].

Recently, important progress was made in the case of independent identically distributed (iid) tensor models. The correlation functions are solved analytically in the large  $N$ -limit, in which the dominant graphs are called “*melon*” [17]. This model corresponds to dynamical triangulations in three and higher dimensions. The susceptibility exponent is computed and the model is reminiscent of certain models of branched polymers [23]. In the continuum limit, the models exhibit two phase transitions. Despite all these aesthetic results, the critical behavior of the large- $N$  limit of the renormalizable models (*the melonic approximation*) is not yet explored. The phase transitions must be computed explicitly. This glimpse needs to be taking into account for the future development of the renormalizable TGFT program.

This paper extends previous work on Schwinger-Dyson equations for matrix and tensor models. The original motivation for this method was the construction of the  $\phi_4^4$ -model on noncommutative Moyal space. The model is perturbatively renormalizable [39]-[41] and asymptotically safe in the UV regime [42]-[43]. The key step of the asymptotic safety proof [43] was extended in [44] to obtain a closed equation for the two-point function of the model. This equation was reduced in [45] to a fixed point problem for which existence of a solution was proved. All higher correlation functions were expressed in terms of the fixed point solution. In [46] the fixed point problem was numerically studied. This gave evidence for phase transitions and for reflection positivity of the Schwinger two-point function.

The noncommutative  $\phi_4^4$ -model solved in [45] can be viewed as the quartic cousin of the Kontsevich model which is relevant for two-dimensional quantum gravity. This leads immediately to the question to extend the techniques of [44, 45] to tensor models of rank  $D \geq 3$ . In [48] one of us addressed the closed equation for correlation functions of rank 3 and 4 just renormalizable TGFT. The two-point functions are given perturbatively using the iteration method. The main challenge in this new direction is to perform the combinatorics of Feynman graphs and to solve the nontrivial integral equations of the correlators. The nonperturbative study of all correlation functions need to be investigated carefully.

In this paper we push further this program. For this, we consider the just renormalizable tensor model of the form  $\varphi_5^4$  without gauge condition, whose dynamics is described by the propagator of the form  $1/\mathbf{p}^2$ . In the melonic approximation, the Schwinger-Dyson equations are given. The closed equation of the two-point and four-point functions are derived and its solution discussed.

The paper is organized as follows. In section 2, proceeding from the definition of the model and its symmetries, we give the Ward-Takahashi identities which result from these symmetries. In section 3 we find the melonic approximation of the Schwinger-Dyson equa-

tion. Section 4 investigates the closed equation for two- and four-point functions. Section 5 is devoted to the closed equation of the four-point correlation functions. In section 6 we solve the equation obtained. In Section 7 we give the conclusions, open questions and future work.

## 2 The Models

The model we will be mainly considering here is a tensorial  $\phi^4$ -theory on  $U(1)^{\times 5}$ . Namely,

$$S[\bar{\varphi}, \varphi] = \int_{U(1)^5} d\mathbf{g} \bar{\varphi}(\mathbf{g})(-\Delta + m^2)\varphi(\mathbf{g}) + \frac{\lambda}{2} \sum_{c=1}^5 \int_{U(1)^{20}} d\mathbf{g} d\mathbf{g}' d\mathbf{h} d\mathbf{h}' \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}')\bar{\varphi}(\mathbf{h})\varphi(\mathbf{h}')K_c(\mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}'), \quad (1)$$

where  $\Delta = \sum_{\ell=1}^5 \Delta_\ell$  and  $\Delta_\ell$  is the Laplace-Beltrami operator on  $U(1)$  acting on colour- $\ell$  indices [32], bold variables stand for 5-dimensional variables ( $\mathbf{g} = (g_1, \dots, g_5)$ ), and  $K_c$  identifies group variables according to a vertex of colour  $c \in \{1, 2, \dots, 5\}$ . Figure 1 shows the vertex of colour 1.

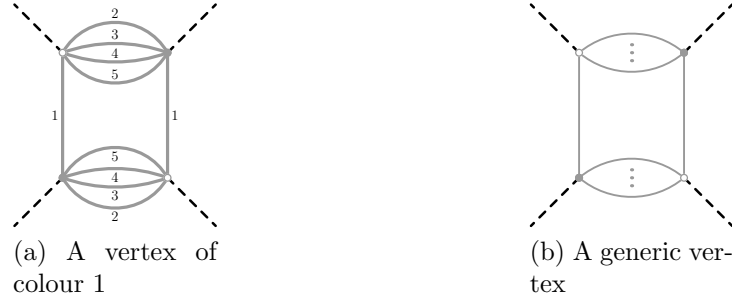


Figure 1: Vertices

The statistical physics description of the model is encoded in the partition function:

$$\mathcal{Z}[\bar{J}, J] = \int D\varphi D\bar{\varphi} e^{-S[\bar{\varphi}, \varphi] + J\bar{\varphi} + \varphi J} = e^{W[\bar{J}, J]}, \quad (2)$$

where  $\bar{J}$  and  $J$  represent the sources and  $W[\bar{J}, J]$  is the generating functional for the connected Green's functions. Then the  $N$ -point Green functions take the form

$$G_N(\mathbf{g}_1, \dots, \mathbf{g}_{2N}) = \frac{\partial \mathcal{Z}(\bar{J}, J)}{\partial J_1 \partial \bar{J}_1 \dots \partial J_N \partial \bar{J}_N} \Big|_{J=\bar{J}=0}. \quad (3)$$

Now let  $\varphi_{class}$  denote the classical field defined by the expectation value of  $\varphi$  in the presence of sources  $J, \bar{J}$ :

$$\varphi_{class} = \langle \varphi \rangle = \frac{\delta W[\bar{J}, J]}{\delta \bar{J}}, \quad \bar{\varphi}_{class} = \langle \bar{\varphi} \rangle = \frac{\delta W[\bar{J}, J]}{\delta J}. \quad (4)$$

Then the 1PI effective action  $\Gamma_{1PI}$  is given by the Legendre transform of  $W[\bar{J}, J]$  as

$$\Gamma_{1PI} = -W[\bar{J}, J] + \int (J\bar{\varphi}_{class} + \varphi_{class}\bar{J}). \quad (5)$$

The correlation functions can be computed perturbatively by expanding the interaction part of the action (1):

$$G_N(\mathbf{g}_1, \dots, \mathbf{g}_{2N}) \sim \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} \int d\mu_C \bar{\varphi}(\mathbf{g}_1) \cdots \varphi(\mathbf{g}_{2N}) \quad (6)$$

$$\times \left[ \sum_{c=1}^5 \int_{U(1)^{20}} d\mathbf{g} d\mathbf{g}' d\mathbf{h} d\mathbf{h}' \bar{\varphi}(\mathbf{g}) \varphi(\mathbf{g}') \bar{\varphi}(\mathbf{h}) \varphi(\mathbf{h}') K_c(\mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}') \right]^n$$

where  $d\mu_C$  is the Gaussian measure with covariance  $C$  i.e:

$$\int d\mu_C \bar{\varphi}(\mathbf{g}) \varphi(\mathbf{g}') = C(\mathbf{g}, \mathbf{g}'), \quad \int d\mu_C \varphi(\mathbf{g}) \varphi(\mathbf{g}') = \int d\mu_C \bar{\varphi}(\mathbf{g}) \bar{\varphi}(\mathbf{g}') = 0. \quad (7)$$

In all of this paper we consider the Fourier transform of the field  $\varphi$  to ‘momentum space’ and write  $\varphi(p_1, \dots, p_5) = \varphi_{12345} = \varphi_{\mathbf{p}}$ , with  $\mathbf{p} \in \mathbb{Z}^5$ . We define a unitary transformation of rank- $D$  tensor fields  $\varphi, \bar{\varphi}$  under the tensor product of  $D$  fundamental representations of the unitary group  $\mathcal{U}_{\otimes}^{N_D} := \otimes_{i=1}^D U(N_i)$ . For  $U^{(a)} \in U(N_a)$ ,  $a = 1, 2, \dots, D$ , we define

$$\varphi_{12\dots D} \rightarrow [U^{(a)}\varphi]_{12\dots a\dots D} = \sum_{p'_a \in \mathbb{Z}} U_{p_a p'_a}^{(a)} \varphi_{12\dots a'\dots D}, \quad (8)$$

$$\bar{\varphi}_{12\dots D} \rightarrow [\bar{\varphi}U^{\dagger(a)}]_{12\dots a\dots D} = \sum_{p'_a \in \mathbb{Z}} \bar{U}_{p_a p'_a}^{(a)} \bar{\varphi}_{12\dots a'\dots D}. \quad (9)$$

Here,  $p'_a$  or simply  $a'$  is the momentum index at the position  $a$  in the expression  $\varphi_{12\dots a'\dots D}$ . For  $N_i = N$ , we choose the interaction terms of (1) in such a way that they are invariant under the transformation  $U^{(a)}$ , i.e.  $\delta^{(a)} S^{\text{int}} = 0$ . Note that the measure  $d\varphi d\bar{\varphi}$  is also invariant under  $U^{(a)}$ . Let us consider now the infinitesimal Hermitian operator corresponding to the generator of unitary group  $U(N_a)$ , i.e.

$$U_{pp'}^{(a)} = \delta_{pp'}^{(a)} + iB_{pp'}^{(a)} + O(B^2), \quad \bar{U}_{pp'}^{(a)} = \delta_{pp'}^{(a)} - i\bar{B}_{pp'}^{(a)} + O(\bar{B}^2), \quad (10)$$

with  $\bar{B}_{pp'}^{(a)} = B_{p'p}^{(a)}$ . Then the variation of the partition function respect to  $B$ , i.e.  $\frac{\delta \ln \mathcal{Z}}{\delta B} = 0$  gives the Ward-Takahashi identities which are written as

$$\sum_{p_2, \dots, p_D} (C_{m2\dots D}^{-1} - C_{n2\dots D}^{-1}) \langle \varphi_{[\alpha]} \bar{\varphi}_{[\beta]} \varphi_{n2\dots D} \bar{\varphi}_{m2\dots D} \rangle_c = \delta_{m\alpha_1} \langle \varphi_{n\alpha_2\dots\alpha_D} \bar{\varphi}_{\beta_1\dots\beta_D} \rangle_c$$

$$- \delta_{n\beta_1} \langle \bar{\varphi}_{m\beta_2\dots\beta_D} \varphi_{\alpha_1\dots\alpha_D} \rangle_c, \quad (11)$$

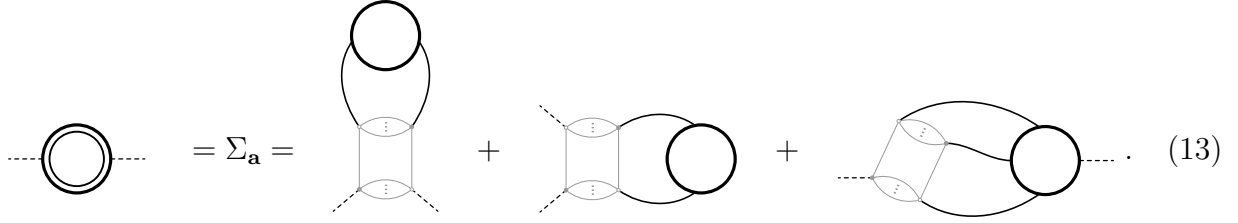
where  $C_{p_1\dots p_D}$  denotes the propagator. For more detail concerning relation (11) see [48]. The correlation functions with insertion of strands are denoted by  $G_{[mn]\dots}^{\text{ins}} = \varphi_{[\alpha]} \bar{\varphi}_{[\beta]} \varphi_{n2\dots D} \bar{\varphi}_{m2\dots D}$ . Then the relation (11) takes form as

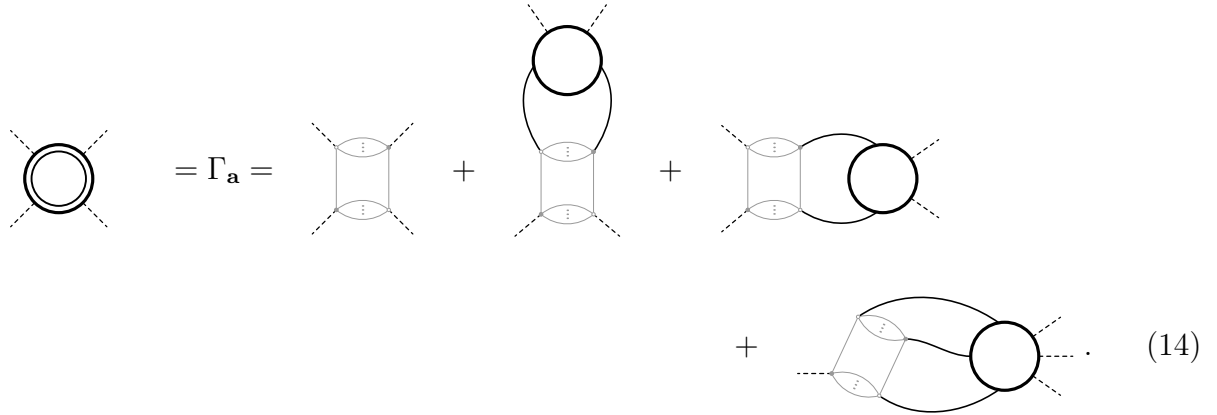
$$\sum_{2,3,\dots,D} (C_{m2\dots D}^{-1} - C_{n2\dots D}^{-1}) G_{[mn]\dots}^{\text{ins}} = G_{n\dots} - G_{m\dots} \quad (12)$$

The model (1) is (just) renormalizable to all orders of perturbation theory. See Refs [27]-[32] for more detail.

### 3 Schwinger-Dyson equation in the melonic approximation

We start by writing the Schwinger-Dyson equations for the one-particle irreducible 2- and 4-point functions of the model (1). We use the following graphical conventions: dashed lines symbolize amputated external legs, a black circle represents a connected function whereas two concentric circles stand for a one-particle irreducible function. Finally, in order to lighten equations, we will use the generic vertex of fig. 1b to mean the sum of the five different coloured interactions. Note that (13) has been derived in [48].





We now want to restrict our attention to the melonic part of the 2- and 4-point functions. Let  $\mathcal{G}$  be a 2- or 4-point Feynman graph of model (1). Let denoted by  $\omega(\mathcal{G})$  the degree of the tensor graph  $\mathcal{G}$  i.e:

$$\omega(\mathcal{G}) = \sum_{J \text{ jacket of } \mathcal{G}} g_J \quad (15)$$

where  $g_J$  is the genus of the jacket  $J$ . We impose  $\omega(\mathcal{G}) = 0$ . We will prove that not all terms of eqs. (13) and (14) contribute to the melonic functions. A simple way of computing the degree  $\omega$  of a graph is to count its number  $F$  of faces. Indeed, the two are related in the following way (in dimension 5, for a degree 4 interaction)[48]:

$$F = 4V + 4 - 2N - \frac{1}{12}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) \quad (16)$$

where  $V$  is the number of vertices of  $\mathcal{G}$ ,  $N$  its number of external legs, and  $C_{\partial\mathcal{G}}$  is the number of connected components of its boundary graph  $\partial\mathcal{G}$  and  $\tilde{\omega}(\mathcal{G}) = \sum_{\tilde{J} \subset \mathcal{G}} g_{\tilde{J}}$  with

$\tilde{J}$  the pinched jacket associated with a jacket  $J$  of  $\mathcal{G}$ . Recall that the Feynman graphs here are so-called uncoloured graphs and, as a consequence, a face is a cycle of colours  $0i$ ,  $i \in \{1, 2, \dots, 5\}$  [11]. A detailed analysis of coloured graphs [30, 48] allows to prove that  $F(\mathcal{G}) = F_{\max}(\mathcal{G}) = 4V + 4 - 2N$ , if and only if  $\tilde{\omega}(\mathcal{G}) = \omega(\partial\mathcal{G}) = C_{\partial\mathcal{G}} - 1 = 0$ . Moreover  $F \leq F_{\max}$ . We can thus prove the following

**Lemma 1.** *The Schwinger-Dyson equations for the melonic 2- and 4-point functions of model (1) are (m stands for melonic):*

$$\text{Diagram (17)} \quad (17)$$

$$\text{Diagram (18)} \quad (18)$$

*Proof.* The right-hand side of eqs. (13) and (14) involve connected 2-, 4-, and 6-point function insertions and a generic vertex. Let  $\mathcal{G}$  be a graph contributing to the left-hand side of (13) or (14) and let  $F$  be its number of faces. Let us study a term of the right-hand side of the equation under consideration. The number of faces of a graph contributing to its insertion is written  $F'$ . Clearly  $F = F' + \delta F$  where  $\delta F \geq 0$ . The additional internal faces are created by closing the external faces of the insertion with the new edges connected to the new vertex. As a consequence,  $\delta F$  is bounded above by the number of faces of the new vertex which do not contain its external legs. Note also that  $F \leq F'_{\max} + \delta F$ .

Let us now consider eq. (13) and the lying tadpole of its right-hand side (second term). In this case,  $\delta F \leq 1$ . From eq. (16),  $F'_{\max} = 4V'$  ( $V'$  being the number of vertices of the connected 2-point insertion) and  $F \leq 4V' + 1 < 4(V' + 1) = F_{\max}$ . Thus whatever the insertion, the graph  $\mathcal{G}$  cannot be melonic. The same type of argument holds for the other terms but for the sake of clarity, let us repeat it for the last term of eq. (14). Here  $\delta F \leq 5$  and  $F'_{\max} = 4V' - 8$ . Their sum never reaches  $F_{\max} = 4V - 4 = 4V'$ .

The only terms which survive this analysis are the first one of eq. (13), and the first and second ones of eq. (14). Moreover it also proves that for a graph to be melonic, the corresponding insertion needs to be melonic too. Note that a melonic graph necessarily has a melonic boundary [28, 34]. Finally, such arguments also fix the orientation, and the colour, of the boundary graph of the 4-point insertion in the second term on the right-hand side of 18, see fig. 2 for a zoom into this term.  $\square$

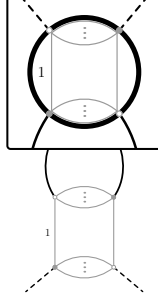


Figure 2: Boundary structure of a melonic 4-point insertion

Note that the Schwinger-Dyson equation (17) and (18) are easy to describe. Taking into account (17) we do not need to write the Ward-Takahashi identities before getting the closed equation of the two-point functions.

## 4 Two-point correlation functions

We now want to use the melonic approximation to obtain a closed equation for the 1PI two-point function  $\Sigma_{a_1, \dots, a_5}$ . For sake of simplicity write  $\mathbf{a} = (a_1, \dots, a_5) \in \mathbb{Z}^5$ . Setting each constant  $\lambda_\rho$  ( $\rho = 1, \dots, 5$ ) equal to the bare coupling constant,  $\lambda_\rho = \lambda$ , we can express the 1PI two-point function  $\Sigma_{\mathbf{a}}$  in terms of the renormalized quantities by using the Taylor expansion

$$\begin{aligned} \Sigma_{\mathbf{a}} &= \Sigma_{\mathbf{0}} + |\mathbf{a}|^2 \frac{\partial \Sigma_{\mathbf{a}}}{\partial |\mathbf{a}|^2} \Big|_{\mathbf{a}=\mathbf{0}} + \Sigma_{\mathbf{a}}^r \\ &= (Z-1)|\mathbf{a}|^2 + Zm^2 - m_r^2 + \Sigma_{\mathbf{a}}^r, \end{aligned} \quad (19)$$

with

$$m^2 = \frac{m_r^2 + \Sigma_{\mathbf{0}}}{Z}, \quad Z = 1 + \frac{\partial \Sigma_{\mathbf{a}}}{\partial |\mathbf{a}|^2} \Big|_{\mathbf{a}=\mathbf{0}}. \quad (20)$$

Moreover the following renormalization conditions

$$\Sigma_{\mathbf{0}}^r = 0, \quad \frac{\partial \Sigma_{\mathbf{a}}^r}{\partial a_\rho^2} \Big|_{\mathbf{a}=\mathbf{0}} = 0 \quad (21)$$

hold.

The propagator  $C$ , given explicitly by  $C_{\mathbf{p}}^{-1} = Z(|\mathbf{p}|^2 + m^2)$ , is related to the dressed propagator  $G_{\mathbf{a}}$  by means of the Dyson relation  $G_{\mathbf{a}}^{-1} = C_{\mathbf{a}}^{-1} - \Sigma_{\mathbf{a}}$ . Then using the

Schwinger-Dyson equations for  $\Sigma_{\mathbf{a}}$ , given in (17), we get

$$\begin{aligned} \Sigma_{\mathbf{a}} = -Z^2 \lambda \sum_{p_1, p_2, p_3, p_4}^{\Lambda} & \left[ \frac{1}{C_{a_1 p_1 p_2 p_3 p_4}^{-1} - \Sigma_{a_1 p_1 p_2 p_3 p_4}} + \frac{1}{C_{p_1 a_2 p_2 p_3 p_4}^{-1} - \Sigma_{p_1 a_2 p_2 p_3 p_4}} \right. \\ & + \frac{1}{C_{p_1 p_2 a_3 p_3 p_4}^{-1} - \Sigma_{p_1 p_2 a_3 p_3 p_4}} + \frac{1}{C_{p_1 p_2 p_3 a_4 p_4}^{-1} - \Sigma_{p_1 p_2 p_3 a_4 p_4}} \\ & \left. + \frac{1}{C_{p_1 p_2 p_3 p_4 a_5}^{-1} - \Sigma_{p_1 p_2 p_3 p_4 a_5}} \right]. \end{aligned} \quad (22)$$

The sums are performed over the integers  $p_i \in \mathbb{Z}$  with some cutoff  $\Lambda$ . For  $\rho = 1, \dots, 5$ , let  $\sigma_\rho$  be the action of  $\mathfrak{S}_5$  which permutes the strands with momenta  $\mathbf{p}$  as follows:

$$\begin{aligned} \sigma_1(p_1 p_2 p_3 p_4 p_5) &= (p_2 p_1 p_3 p_4 p_5), \\ \sigma_2(p_1 p_2 p_3 p_4 p_5) &= (p_2 p_3 p_1 p_4 p_5), \\ &\vdots \\ \sigma_4(p_1 p_2 p_3 p_4 p_5) &= (p_2 p_3 p_4 p_5 p_1), \end{aligned}$$

and  $\sigma_5$  trivially. Notice that the value of the propagator  $C_{\mathbf{p}}$  remains invariant under the action of all these  $\sigma_\rho$ ,  $C_{\sigma_\rho(\mathbf{p})} = C_{\mathbf{p}}$ . After combining (19) and (22) we can obtain, by using

$$C_{a_1 p_1 p_2 p_3 p_4}^{-1} - \Sigma_{a_1 p_1 p_2 p_3 p_4} = a_1^2 + \sum_{i=1}^4 p_i^2 + m_r^2 - \Sigma_{a_1 p_1 p_2 p_3 p_4}^r, \quad (23)$$

the expression

$$(Z-1)|\mathbf{a}|^2 + Zm^2 - m_r^2 + \Sigma_{\mathbf{a}}^r = -Z^2 \lambda \sum_{\rho=1}^5 \sum_{p_1 p_2 p_3 p_4}^{\Lambda} \frac{1}{(a_\rho^2 + \sum_{i=1}^4 p_i^2) + m_r^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^r}. \quad (24)$$

We now can evaluate at  $\mathbf{a} = \mathbf{0}$  to get rid of the term  $Zm^2 - m_r^2$ , which according to this equation is given by

$$Zm^2 - m_r^2 = -Z^2 \lambda \sum_{p_1 p_2 p_3 p_4}^{\Lambda} \sum_{\rho=1}^5 \frac{1}{\sum_{i=1}^4 p_i^2 + m_r^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^r}. \quad (25)$$

Replacing the expression (25) in (24), we obtain

$$\begin{aligned} (Z-1)|\mathbf{a}|^2 + \Sigma_{\mathbf{a}}^r = -Z^2 \lambda \sum_{p_1 p_2 p_3 p_4}^{\Lambda} \sum_{\rho=1}^5 & \left[ \frac{1}{a_\rho^2 + |\mathbf{p}|^2 + m_r^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^r} \right. \\ & \left. - \frac{1}{|\mathbf{p}|^2 + m_r^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^r} \right]. \end{aligned} \quad (26)$$



Here we have defined  $|\mathbf{p}|^2 := \sum_{i=1}^4 p_i^2$ , with some abuse of notation. The evaluation at  $\mathbf{a} = \sigma_\rho(a_\rho 0000)$ , namely

$$(Z-1)a_\rho^2 + \Sigma_{\sigma_\rho(a_\rho 0000)}^r = -Z^2\lambda \sum_{\mathbf{p} \in \mathbb{Z}^4}^\Lambda \left[ \frac{1}{a_\rho^2 + |\mathbf{p}|^2 + m_r^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^r} - \frac{1}{|\mathbf{p}|^2 + m_r^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^r} \right], \quad (27)$$

leads to a splitting of the renormalized 1PI two-point function as

$$\Sigma_{a_1 a_2 a_3 a_4 a_5}^r = \sum_{\rho=1}^5 \Sigma_{\sigma_\rho(a_\rho 0000)}^r \quad (28)$$

as a mere consequence of summing eq. (27) over  $\rho = 1, \dots, 5$  and then comparing the rhs of the resulting equation with that of eq. (26). The wave function renormalization constant  $Z$  can be obtained from differentiating (27) with respect to any  $a_\rho^2$  and the subsequent evaluation at  $a_\rho = 0$ :

$$Z-1 = Z^2\lambda \left[ \sum_{\mathbf{p} \in \mathbb{Z}^4}^\Lambda \frac{1}{(|\mathbf{p}|^2 + m_r^2 - \Sigma_{0\mathbf{p}}^r)^2} \right], \quad \Lambda \in \mathbb{Z}^4. \quad (29)$$

Here (21) has been used. Insertion of this value for  $(Z-1)$  into eq. (27) renders, setting  $\tilde{\lambda} = Z^2\lambda$  and using (28) again,

$$\Sigma_{a0}^r = -\tilde{\lambda} \sum_{\mathbf{p} \in \mathbb{Z}^4}^\Lambda \left[ \frac{1}{a^2 + |\mathbf{p}|^2 + m_r^2 - \Sigma_{a0}^r - \Sigma_{0\mathbf{p}}^r} + \frac{a^2}{(|\mathbf{p}|^2 + m_r^2 - \Sigma_{0\mathbf{p}}^r)^2} - \frac{1}{|\mathbf{p}|^2 + m_r^2 - \Sigma_{0\mathbf{p}}^r} \right]. \quad (30)$$

The above equation could lead to a divergence in the limit where  $\Lambda \rightarrow \infty$  which should compensate with a divergence of  $\tilde{\lambda}^{-1}$ . We will prove this in sec. 5.

We now pass to a continuum limit in which the discrete momenta  $a \in \mathbb{Z}, \mathbf{p} \in \mathbb{Z}^4$  become continuous. We do this here in a formal manner. A rigorous treatment should first view the regularized dual of  $U(1)^5$  as a toroidal lattice  $(\mathbb{Z}/2\Lambda\mathbb{Z})^5$ , then take an appropriate scaling limit to the 5-torus  $[-\Lambda, \Lambda]^5$  with periodic boundary conditions, and finally  $\Lambda \rightarrow \infty$ . These steps should give for (30):

$$\Sigma_{a0}^r = -\tilde{\lambda} \int_{\mathbb{R}^4} d\mathbf{p} \left[ \frac{a^2}{(|\mathbf{p}|^2 + m_r^2 - \Sigma_{0\mathbf{p}}^r)^2} + \frac{1}{a^2 + |\mathbf{p}|^2 + m_r^2 - \Sigma_{a0}^r - \Sigma_{0\mathbf{p}}^r} - \frac{1}{|\mathbf{p}|^2 + m_r^2 - \Sigma_{0\mathbf{p}}^r} \right] \quad (31)$$

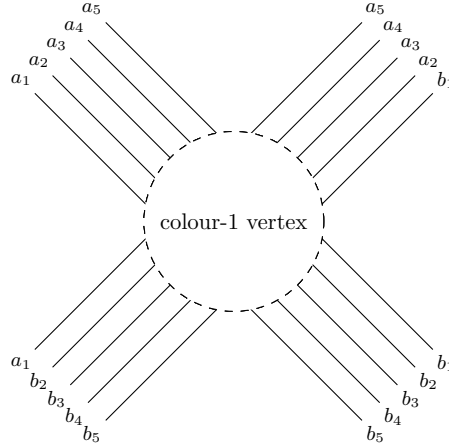
with  $d\mathbf{p} = dp_1 dp_2 dp_3 dp_4$ . Because of (28), i.e.  $\Sigma_{0\mathbf{p}}^r = \sum_{i=1}^4 \Sigma_{p_i 0}^r$ , (31) is a closed equation for the function  $\Sigma_{a0}^r$ . Using Taylor's formula we can equivalently write this equation as

$$\Sigma_{a0}^r = -\tilde{\lambda} \int_0^{a^2} dt (a^2 - t) \int_{\mathbb{R}^4} d\mathbf{p} \frac{d^2}{dt^2} \left( \frac{1}{m_r^2 + t - \Sigma_{\sqrt{t}0}^r + \sum_{i=1}^4 (p_i^2 - \Sigma_{p_i 0}^r)} \right). \quad (32)$$

The equation (32) is the analogue of the fixed point equation [45, eq. (4.48)] for the boundary 2-point function  $G_{a0}$  of the quartic matrix model: In both situations the decisive function satisfies a non-linear integral equation for which we can at best expect an approximative numerical solution. Finding a suitable method, implementing it in a computer program and running the computation needs time. We intend to report results in a future publication. At the moment we have to limit ourselves to a perturbative investigation of this equation, see sec. 6.

## 5 Closed equation of the 1PI four-point functions

In this section we prove that the coupling constant  $\tilde{\lambda}$  is finite in the  $UV$  regime. It will be convenient to briefly discuss first the index structure of the four-point function.  $\Gamma^4$  has 10 indices: Each external coloured line of  $\varphi_{\mathbf{a}}$  and  $\varphi_{\mathbf{b}}$  should be paired with one of the complex conjugate fields  $\bar{\varphi}_{\mathbf{c}}$  and  $\bar{\varphi}_{\mathbf{d}}$  in the vertex  $\varphi_{\mathbf{a}}\bar{\varphi}_{\mathbf{c}}\varphi_{\mathbf{b}}\bar{\varphi}_{\mathbf{d}}$ . That is to say that  $\mathbf{c}$  and  $\mathbf{d}$  are expressed<sup>1</sup> in terms of  $(\mathbf{a}, \mathbf{b})$ . For instance, for the vertex of colour 1, represented in fig. (1a),  $\mathbf{c} = (a_5 a_4 a_3 a_2 b_1)$ , and  $\mathbf{d} = (b_5 b_4 b_3 b_2 a_1)$ . The external lines for that vertex look as follows:



We now excise the vertex in the rhs of the melonic approximation of the Schwinger-Dyson equation for the four-point function and write its value,  $-Z^2\lambda$ , instead. The first graph in the rhs of eq. (18) is precisely the vertex. In the second graph, after removing the vertex, a jump in the colour 1 occurs; this can be understood as an insertion, whose value we give now. The removal of the colour-1 vertex in that graph leaves the following graph, where the

<sup>1</sup>More precisely,  $\mathbf{c} = (\pi_1 \circ \varrho)(\mathbf{a}, \mathbf{b})$  and  $\mathbf{d} = (\pi_2 \circ \varrho)(\mathbf{a}, \mathbf{b})$  where,  $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{10}$ ,  $\pi_1$  and  $\pi_2$  are the projections in the first or second factor of  $\mathbb{Z}^5 \oplus \mathbb{Z}^5$ , and  $\varrho$  is a permutation in  $\mathfrak{S}_{10}$  that allows colour conservation.

upper dotted lines have indices  $\mathbf{a} = (a_1 a_2 a_3 a_4 a_5)$  and  $\mathbf{c} = (a_5 a_4 a_3 a_2 b_1)$ .

$$G_{a_1 a_2 a_3 a_4 a_5}^{-1} G_{a_5 a_4 a_3 a_2 b_1}^{-1} G_{[a_1 b_1] a_2 a_3 a_4 a_5}^{\text{ins}} = \text{Diagram} \quad . \quad (33)$$

According to (12), the value of that insertion is

$$G_{[a_1 b_1] a_2 a_3 a_4 a_5}^{\text{ins}} = \frac{1}{Z(a_1^2 - b_1^2)} (G_{a_1 a_2 a_3 a_4 a_5} - G_{a_5 a_4 a_3 a_2 b_1}).$$

In general any of the vertex in this model has a privileged colour  $i$  (i.e. the colour  $i$  is with the neighbour vertically and the remaining colours are connected with the other neighbouring field, sideways). The excised graph for the ‘colour  $i$ ’-vertex has then the following value:

$$G_{a_1 a_2 a_3 a_4 a_5}^{-1} G_{a_5 \dots b_1 \hat{a}_i \dots a_1}^{-1} G_{[a_i b_i] a_1 \dots \hat{a}_i \dots a_5}^{\text{ins}} = \frac{1}{Z(a_i^2 - b_i^2)} \left[ \frac{1}{G_{a_5 \dots b_i \hat{a}_i \dots a_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right],$$

where  $\hat{a}$  means omission of  $\hat{a}_i$  (and this index is substituted by  $b_i$ ) and, accordingly,  $\mathbf{c} = (a_5 \dots b_i \hat{a}_i \dots a_2 a_1)$ . Then the full equation for  $\Gamma_{a_1 a_2 a_3 a_4 a_5 b_1 b_2 b_3 b_4 b_5}^{4, \text{ren}}$  is given by the sum over these two kinds of graphs over all the vertices of the model, to wit

$$\begin{aligned} \Gamma_{\mathbf{a}, \mathbf{b}}^{4, \text{ren}} &= \sum_{i=1}^5 -Z^2 \lambda (1 + G_{a_1 a_2 a_3 a_4 a_5}^{-1} G_{a_5 \dots b_1 \hat{a}_i \dots a_1}^{-1} G_{[a_i b_i] a_1 \dots \hat{a}_i \dots a_5}^{\text{ins}}) \\ &= -Z^2 \lambda \left( 5 + \frac{1}{Z(a_1^2 - b_1^2)} \left[ \frac{1}{G_{a_5 a_4 a_3 a_2 b_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right] \right. \\ &\quad + \frac{1}{Z(a_2^2 - b_2^2)} \left[ \frac{1}{G_{a_5 a_4 a_3 b_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right] + \frac{1}{Z(a_3^2 - b_3^2)} \left[ \frac{1}{G_{a_5 a_4 b_3 a_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right] \\ &\quad \left. + \frac{1}{Z(a_4^2 - b_4^2)} \left[ \frac{1}{G_{a_5 b_4 a_3 a_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right] + \frac{1}{Z(a_5^2 - b_5^2)} \left[ \frac{1}{G_{b_5 a_4 a_3 a_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right] \right). \end{aligned}$$

By inserting the value for  $G_{\mathbf{q}}$  given by (23), and by imposing the renormalization conditions, taking the limit  $\mathbf{a}, \mathbf{b} \rightarrow 0$  one readily obtains

$$\Gamma_0^{4, \text{ren}} = -5\tilde{\lambda} \left( 1 + \frac{1}{Z} \right). \quad (34)$$

By imposing the cutoff  $\Lambda$ , we can show (perturbatively) that the wave function renormalization (for a similar computation cf. Lemma 5 in [30]) takes the form

$$Z = 1 + x\lambda \log(\Lambda) + \mathcal{O}(\lambda^2), \quad x \in \mathbb{R}. \quad (35)$$

Then one has

$$-\lambda_r = \Gamma_0^{\text{ren}} \rightarrow -5\tilde{\lambda}. \quad (36)$$

## 6 Solution of the integral equation

The integral equation (32) is a non-linear integro-partial differential equation. We therefore opt for a numerical approach. We introduce the following dimensionless variables:

$$\alpha \equiv \frac{a}{m_r}, \quad \tau \equiv \frac{t}{m_r^2}, \quad \boldsymbol{\rho} \equiv \frac{\mathbf{p}}{m_r}, \quad \text{and} \quad \gamma \equiv 1 + \tau + \sum_{i=1}^4 \rho_i^2,$$

and, accordingly, we rescale the two-point function  $\sigma(\alpha) \equiv \Sigma_{a0000}^r/m_r^2$ . Equation (32) can be thus rewoded:

$$\sigma(\alpha) = -\tilde{\lambda} \int d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{1 + \tau + |\boldsymbol{\rho}|^2 - \sigma(\sqrt{\tau}, \boldsymbol{\rho})} \right\}. \quad (37)$$

Expanding the solution in  $\tilde{\lambda}$ ,  $\sigma(\alpha) = \sum_{n=0}^{\infty} \sigma_n(\alpha) \tilde{\lambda}^n$ , it readily follows  $\sigma_0(\alpha) = 0$ . To proceed with the computation of the non-trivial orders, we invert the power series (in  $\tilde{\lambda}$ ) appearing in the denominator (37) after factoring out  $\gamma$ , namely  $(1 - \sigma(\sqrt{\tau}, \boldsymbol{\rho})/\gamma)$ . First, we treat this series as a formal power series, then we care about convergence. The idea is that in order to compute  $\sigma_{n+1}$ , for which we need  $\sigma_i$ ,  $i \leq n$ , we approximate the latter functions by near-to-‘principal diagonal’ Padé approximants, i.e. by quotients of polynomials of almost equal degree; this approximation is valid in a certain domain and would lead to the convergence of the series there. Shortly, a second advantage of the Padé approximants will be evident.

We use the following result for the power of a series (cf. sec. 3.5 in [47]): For any  $r \in \mathbb{C}$ , the  $r$ -th power of a formal power series  $1 + g_1 t + \frac{1}{2!} g_2 t^2 + \dots$  can be expanded as follows:

$$\left( 1 + \sum_{n \geq 1} g_n \frac{t^n}{n!} \right)^r = 1 + \sum_{n \geq 1} \left( \mathbb{P}_n^{(r)} \frac{t^n}{n!} \right), \quad (38)$$

where the  $\mathbb{P}_n^{(r)}$ , the so-called *potential polynomials*, are given in terms of the Bell polynomials  $\mathbb{B}_{p,q}$ :

$$\begin{aligned} \mathbb{P}_n^{(r)} &= \sum_{1 \leq k \leq n} (r)_k \mathbb{B}_{n,k}(g_1, \dots, g_{n-k+1}) \\ &= \sum_{1 \leq k \leq n} (-1)^k k! \mathbb{B}_{n,k}(g_1, \dots, g_{n-k+1}). \end{aligned}$$

In our case, the Pochhammer symbol appearing there,  $(r)_k$ , becomes  $(-1)_k = (-1)^k k!$ . As for the Bell polynomials, they are defined by

$$\mathbb{B}_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{c_j} \frac{n!}{c_1! c_2! \dots c_{n-k+1}!} \left( \frac{x_1}{1!} \right)^{c_1} \left( \frac{x_2}{2!} \right)^{c_2} \dots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{c_{n-k+1}}.$$

The sum here runs over all the non-negative integers  $c_l$  such that the conditions

$$\sum_{i=1}^{n-k+1} c_i = k \quad \text{and} \quad \sum_{q=1}^{n-k+1} q c_q = n \quad (39)$$

are fulfilled. It will be useful to rescale the  $k$ -th variable  $x_k$  in the Bell polynomials by  $x'_k = w(k!)x_k$ , for a number  $w \neq 0$ , to obtain a simpler expression in the lhs:

$$\begin{aligned}\mathbb{B}_{n,k}(x'_1, \dots, x'_{n-k+1}) &= \mathbb{B}_{n,k}(w(1!)x_1, w(2!)x_2, \dots, w(n-k+1!)x_{n-k+1}) \\ &= w^k \sum_{c_j} \frac{n!}{c_1!c_2! \dots c_{n-k+1}!} x_1^{c_1} \dots (x_{n-k+1})^{c_{n-k+1}}.\end{aligned}\quad (40)$$

**Remark.** After taking the reciprocal of the power series, the convergence of each coefficient of  $\tilde{\lambda}^n$ ,  $\sigma_n(\alpha)$ , is not guaranteed. We denote by  $\tilde{\sigma}_n(\alpha)$  those probably divergent coefficients, which need to be renormalized. Thus, taking the  $(n+1)$ -order in  $\tilde{\lambda}$  of  $\tilde{\sigma}(\alpha)$ ,  $\tilde{\sigma}_{n+1}(\alpha)$ , boils down to integrate

$$\begin{aligned}\tilde{\sigma}_{n+1}(\alpha) &= - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \\ &\quad \left[ \frac{1}{n!\gamma} \sum_{1 \leq k \leq n} (-1)^k k! \mathbb{B}_{n,k} \left( -1! \frac{\sigma_1(\zeta)}{\gamma}, -2! \frac{\tilde{\sigma}_2(\zeta)}{\gamma}, \dots, -(n-k+1)! \frac{\tilde{\sigma}_{n-k+1}(\zeta)}{\gamma} \right) \right] \\ &= - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left[ \sum_{1 \leq k \leq n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \left( \frac{\tilde{\sigma}_j(\zeta)^{c_j}}{c_j!} \right) \right].\end{aligned}\quad (41)$$

Here  $\zeta = (\sqrt{\tau}, \boldsymbol{\rho})$  and we have made use of (40) with the nowhere-vanishing  $w = -\gamma^{-1}$ . To obtain expressions for higher-order solutions we use the explicit form of the Bell polynomials

$$\begin{aligned}\mathbb{B}_{1,1}(x_1) &= x_1, & \mathbb{B}_{2,1}(x_1, x_2) &= x_2, & \mathbb{B}_{3,1}(x_1, x_2, x_3) &= x_3, & \mathbb{B}_{4,1}(x_1, x_2, x_3, x_4) &= x_4, \\ \mathbb{B}_{2,2}(x_1, x_2) &= x_1^2, & \mathbb{B}_{3,2}(x_1, x_2, x_3) &= 3x_1x_2, & \mathbb{B}_{4,2}(x_1, x_2, x_3, x_4) &= 4x_1x_3 + 3x_2^2, \\ & & \mathbb{B}_{3,3}(x_1, x_2, x_3) &= x_1^3, & \mathbb{B}_{4,3}(x_1, x_2, x_3, x_4) &= 6x_1^2x_2, \\ & & & & \mathbb{B}_{4,4}(x_1, x_2, x_3, x_4) &= x_1^4.\end{aligned}$$

The first order in perturbation theory can be given exactly —and without using the Padé approximants, nor regularization — and is given by

$$\begin{aligned}\sigma_1(\alpha) &= -2\text{Vol}(\mathbb{S}^3) \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \int_0^\infty d\rho \frac{\rho^2}{(1 + \tau + \rho^3)^2} \\ &= -2(2\pi^2) \int_0^{\alpha^2} d\tau \frac{\alpha^2 - \tau}{4(1 + \tau)} = -\pi^2[(\alpha^2 + 1) \log(\alpha^2 + 1) - \alpha^2].\end{aligned}$$

With (41) in our hands, other low-order terms can be obtained:

$$\begin{aligned}
\tilde{\sigma}_0(\alpha) &= \sigma_0(\alpha) = 0 \\
\tilde{\sigma}_1(\alpha) &= \sigma_1(\alpha) = -\pi^2[(\alpha^2 + 1)\log(\alpha^2 + 1) - \alpha^2] \\
\tilde{\sigma}_2(\alpha) &= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left( \frac{1}{\gamma^2} \tilde{\sigma}_1(\zeta) \right) \\
\tilde{\sigma}_3(\alpha) &= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^3} (\tilde{\sigma}_1^2(\zeta) + \gamma \tilde{\sigma}_2(\zeta)) \right\} \\
\tilde{\sigma}_4(\alpha) &= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^4} (\tilde{\sigma}_1^3(\zeta) + 2\gamma \tilde{\sigma}_1(\zeta) \tilde{\sigma}_2(\zeta) + \gamma^2 \tilde{\sigma}_3(\zeta)) \right\} \\
\tilde{\sigma}_5(\alpha) &= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^5} (\tilde{\sigma}_1^4(\zeta) + 3\gamma \tilde{\sigma}_1(\zeta)^2 \tilde{\sigma}_2(\zeta) \right. \\
&\quad \left. + 2\gamma^2 (\tilde{\sigma}_1(\zeta) \tilde{\sigma}_3(\zeta) + \tilde{\sigma}_2^2(\zeta)) + \gamma^3 \tilde{\sigma}_4(\zeta)) \right\}.
\end{aligned}$$

In all these expressions  $\tilde{\sigma}_i(\zeta) = \sum_{j=1}^4 \tilde{\sigma}_i(p_j) + \tilde{\sigma}_i(\sqrt{\tau})$ , with  $\zeta_0 = \sqrt{\tau}, \zeta_1 = \rho_1, \dots, \zeta_4 = \rho_4$ . Notice that the non-linearity is evident from the third order on.

To shed some light on the procedure to extract the divergence occurring in the integral (41), we consider the second order and then extend the method to higher orders. The most dangerous term in

$$\tilde{\sigma}_2(\alpha) = - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[ \frac{6\tilde{\sigma}_1(\zeta)}{\gamma^4} - \frac{4\tilde{\sigma}'_1(\sqrt{\tau})}{\gamma^3} + \frac{\tilde{\sigma}''_1(\sqrt{\tau})}{\gamma^2} \right] \quad (42)$$

is the last summand. We write the Taylor expansion of  $\gamma^{-2}\tilde{\sigma}''_1(\sqrt{\tau})$  at first order and get the renormalized expression  $\sigma_2(\alpha)$  as

$$\begin{aligned}
\sigma_2(\alpha) &= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[ \frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma'_1(\sqrt{\tau})}{\gamma^3} + \frac{\sigma''_1(\sqrt{\tau})}{\gamma^2} - \frac{\sigma''_1(\sqrt{\tau})}{(1 + |\boldsymbol{\rho}|^2)^2} \right] \\
&= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[ \frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma'_1(\sqrt{\tau})}{\gamma^3} \right] \\
&\quad + \pi^2 \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \sigma''_1(\sqrt{\tau}) \log(1 + \tau) \\
&= - \int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[ \frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma'_1(\sqrt{\tau})}{\gamma^3} \right] \\
&\quad + \pi^4 \left\{ (1 + \alpha^2) \log(1 + \alpha^2) - \alpha^2 - \frac{1}{2}(1 + \alpha^2) [\log(1 + \alpha^2)]^2 \right\}.
\end{aligned}$$

The above integral is convergent and therefore  $\sigma_2(\alpha)$  is well defined in the limit where  $\Lambda \rightarrow \infty$ . Consider now

$$\tilde{\sigma}_{n+1}(\alpha) = - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left[ \sum_{1 \leq k \leq n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \left( \frac{\tilde{\sigma}_j(\zeta)^{c_j}}{c_j!} \right) \right]. \quad (43)$$

The integral leads to the logarithmically divergence which could be removed. We get

$$\sigma_{n+1}(\alpha) = - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left\{ \frac{\partial^2}{\partial \tau^2} \left[ \sum_{1 \leq k \leq n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \left( \frac{\sigma_j(\zeta)^{c_j}}{c_j!} \right) \right] - \frac{\sigma_n''(\sqrt{\tau})}{(1 + |\boldsymbol{\rho}|^2)^2} \right\}. \quad (44)$$

The above integral is convergent in the limit where  $\Lambda \rightarrow \infty$  using (almost) equal degree Padé approximation. The solution of the integral equation, for small values of the coupling constant, is given in fig. 3, fig. 4 and fig. 5. Those plots show  $\sigma(\alpha)$ , computed to second order in  $\tilde{\lambda}$ . We have used *Mathematica*<sup>TM</sup> to obtain the Padé approximants and to plot the solution. Their advantage over partial Taylor sums to approximate the  $\sigma_i$ 's becomes now clear— those had been otherwise divergent and the only term we introduced in order to control the divergence would not have been enough.

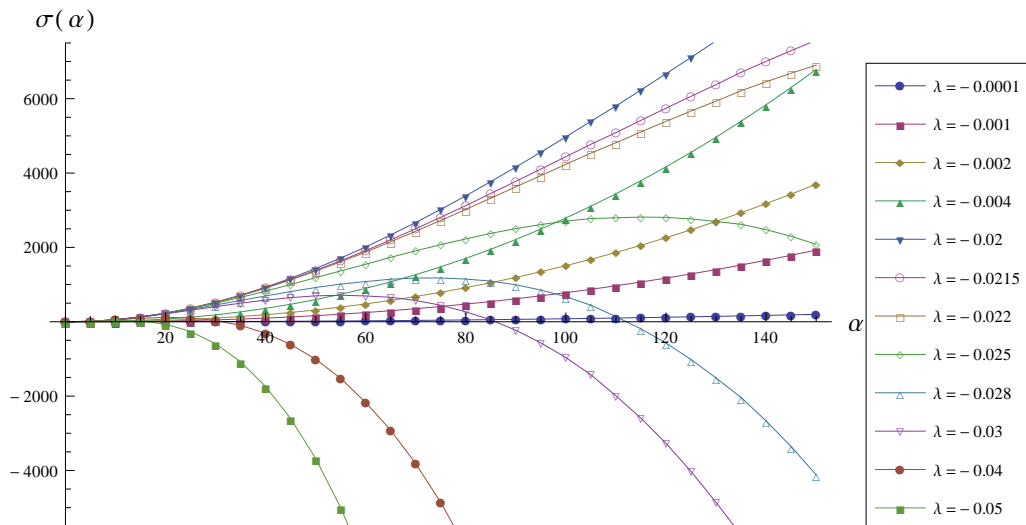


Figure 3: Plot of  $\sigma(\alpha)$  with different negative values of  $\lambda$ . The curves are interpolations of discrete data obtained for the two-point function of the  $\varphi_5^4$ -model (with  $m_r$  set to 1) to second order in  $\tilde{\lambda}$ .

## 7 Conclusion

In this paper we have considered the just renormalizable  $\varphi_5^4$  tensorial group field theory with the propagator of the form  $1/\mathbf{p}^2$ . We have introduced the melonic approximation of the Schwinger-Dyson equation of the two and four-point functions. This made possible, by suppressing the non-melonic graphs, to obtain a closed equation for the two-point functions. This equation is solved perturbatively. It would be interesting to apply the melonic approximation to other tensor models supporting a large- $N$  expansion, e.g. to multi-orientable tensor models [16].

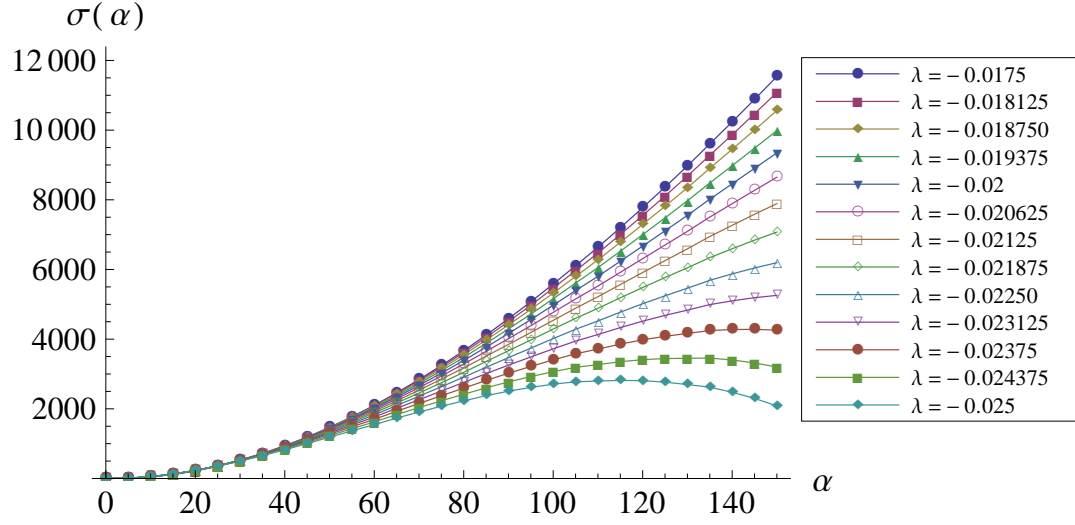


Figure 4: This is a zoom to the region where criticality might take place. It shows how the behaviour of the two-point function bifurcates from a certain value for the coupling constant about  $\lambda \approx -0.002125$ .

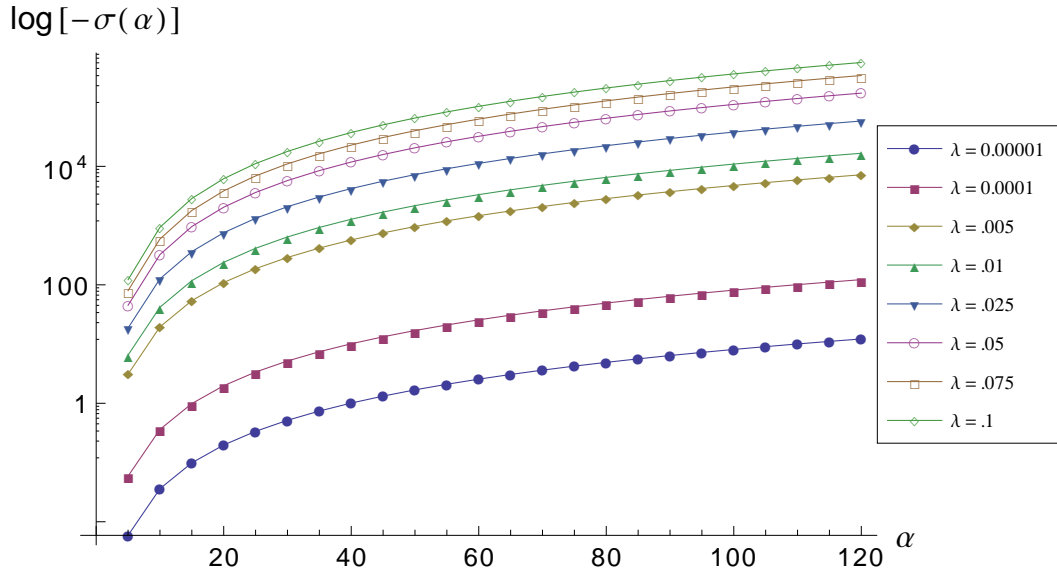


Figure 5: Plot of  $\log[-\sigma(\alpha)]$  with different positive values of  $\lambda$ . Just as in the previous plot, we interpolated a discrete graph.



For future investigation remains the numerical study of the four-point function we treated in section 5. We also plan to address the criticality of the model. Concretely, at certain value of the coupling constant, namely about  $\lambda \approx -2.125 \times 10^{-2}$ , the behaviour of the two-point function noticeably bifurcates. Thus, some criticality is promissory in fig. 4. To claim this we need a new, more detailed study, though; for instance, by solving for higher values of  $\alpha$ . The phase transitions and the critical behaviour of the model could be physically relevant, and in particular, interesting for applications in cosmology.

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## References

- [1] V. Rivasseau, “The Tensor Theory Space,” arXiv:1407.0284 [hep-th].
- [2] V. Rivasseau, “The Tensor Track, III,” Fortsch. Phys. **62**, 81 (2014) [arXiv:1311.1461 [hep-th]].
- [3] V. Rivasseau, “The Tensor Track: an Update,” arXiv:1209.5284 [hep-th].
- [4] V. Rivasseau, “Quantum Gravity and Renormalization: The Tensor Track,” AIP Conf. Proc. **1444**, 18 (2011) [arXiv:1112.5104 [hep-th]].
- [5] D. Oriti, “The Group field theory approach to quantum gravity: Some recent results,” arXiv:0912.2441 [hep-th].
- [6] R. Gurau, “The complete  $1/N$  expansion of colored tensor models in arbitrary dimension,” Annales Henri Poincaré **13**, 399 (2012) [arXiv:1102.5759 [gr-qc]].
- [7] R. Gurau and V. Rivasseau, “The  $1/N$  expansion of colored tensor models in arbitrary dimension,” Europhys. Lett. **95**, 50004 (2011) [arXiv:1101.4182 [gr-qc]].
- [8] R. Gurau, “The  $1/N$  expansion of colored tensor models,” Annales Henri Poincaré **12**, 829 (2011) [arXiv:1011.2726 [gr-qc]].
- [9] R. Gurau and V. Rivasseau, “The  $1/N$  expansion of colored tensor models in arbitrary dimension,” Europhys. Lett. **95**, 50004 (2011) [arXiv:1101.4182 [gr-qc]].

- [10] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” *SIGMA* **8**, 020 (2012) [arXiv:1109.4812 [hep-th]].
- [11] V. Bonzom, R. Gurau and V. Rivasseau, “Random tensor models in the large N limit: Uncoloring the colored tensor models,” *Phys. Rev. D* **85**, 084037 (2012) [arXiv:1202.3637 [hep-th]].
- [12] R. Gurau, “The  $1/N$  Expansion of Tensor Models Beyond Perturbation Theory,” *Commun. Math. Phys.* **330**, 973 (2014) [arXiv:1304.2666 [math-ph]].
- [13] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” *Phys. Rept.* **254**, 1 (1995) [arXiv:hep-th/9306153].
- [14] G. 't Hooft, “On the Convergence of Planar Diagram Expansions,” *Commun. Math. Phys.* **86**, 449 (1982).
- [15] R. Gurau, “The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders,” *Nucl. Phys. B* **865**, 133 (2012) [arXiv:1203.4965 [hep-th]].
- [16] S. Dartois, V. Rivasseau and A. Tanasa “The  $1/N$  expansion of multi-orientable random tensor models,” *Annales Henri Poincare* **15**, 965 (2014) [arXiv:1301.1535 [hep-th]].
- [17] V. Bonzom, R. Gurau, A. Riello and V. Rivasseau, “Critical behavior of colored tensor models in the large N limit,” *Nucl. Phys. B* **853**, 174 (2011) [arXiv:1105.3122 [hep-th]].
- [18] S. Dartois, R. Gurau and V. Rivasseau, “Double Scaling in Tensor Models with a Quartic Interaction,” *JHEP* **1309**, 088 (2013) [arXiv:1307.5281 [hep-th]].
- [19] V. Bonzom, R. Gurau, J. P. Ryan and A. Tanasa, “The double scaling limit of random tensor models,” arXiv:1404.7517 [hep-th].
- [20] R. Gurau, “The Double Scaling Limit in Arbitrary Dimensions: A Toy Model,” *Phys. Rev. D* **84**, 124051 (2011) [arXiv:1110.2460 [hep-th]].
- [21] W. Kaminski, D. Oriti and J. P. Ryan, “Towards a double-scaling limit for tensor models: probing sub-dominant orders,” *New J. Phys.* **16**, 063048 (2014) [arXiv:1304.6934 [hep-th]].
- [22] V. Bonzom, R. Gurau and V. Rivasseau, “The Ising Model on Random Lattices in Arbitrary Dimensions,” *Phys. Lett. B* **711**, 88 (2012) [arXiv:1108.6269 [hep-th]].
- [23] D. Benedetti and R. Gurau, “Phase Transition in Dually Weighted Colored Tensor Models,” *Nucl. Phys. B* **855**, 420 (2012) [arXiv:1108.5389 [hep-th]].
- [24] A. Baratin, S. Carrozza, D. Oriti, J. Ryan and M. Smerlak, “Melonic phase transition in group field theory,” *Lett. Math. Phys.* **104**, 1003 (2014) [arXiv:1307.5026 [hep-th]].
- [25] R. Gurau, “A generalization of the Virasoro algebra to arbitrary dimensions,” *Nucl. Phys. B* **852**, 592 (2011) [arXiv:1105.6072 [hep-th]].

- [26] J. Ben Geloun and V. Bonzom, “Radiative corrections in the Boulatov-Ooguri tensor model: The 2-point function,” *Int. J. Theor. Phys.* **50**, 2819 (2011) [arXiv:1101.4294 [hep-th]].
- [27] J. Ben Geloun and V. Rivasseau, “A Renormalizable 4-Dimensional Tensor Field Theory,” *Commun. Math. Phys.* **318**, 69 (2013) [arXiv:1111.4997 [hep-th]].
- [28] J. Ben Geloun and V. Rivasseau, “Addendum to ‘A Renormalizable 4-Dimensional Tensor Field Theory’,” *Commun. Math. Phys.* **322**, 957 (2013) [arXiv:1209.4606 [hep-th]].
- [29] J. Ben Geloun, “Renormalizable Models in Rank  $d \geq 2$  Tensorial Group Field Theory,” arXiv:1306.1201 [hep-th].
- [30] J. Ben Geloun and D. Ousmane Samary, “3D Tensor Field Theory: Renormalization and One-loop  $\beta$ -functions,” *Annales Henri Poincaré* **14**, 1599 (2013) [arXiv:1201.0176 [hep-th]].
- [31] J. Ben Geloun and R. Toriumi, “Parametric Representation of Rank  $d$  Tensorial Group Field Theory: Abelian Models with Kinetic Term  $\sum_s |p_s| + \mu$ ,” arXiv:1409.0398 [hep-th].
- [32] S. Carrozza, D. Oriti and V. Rivasseau, “Renormalization of Tensorial Group Field Theories: Abelian  $U(1)$  Models in Four Dimensions,” *Commun. Math. Phys.* **327**, 603 (2014) [arXiv:1207.6734 [hep-th]].
- [33] J. Ben Geloun and E. R. Livine, “Some classes of renormalizable tensor models,” *J. Math. Phys.* **54**, 082303 (2013) [arXiv:1207.0416 [hep-th]].
- [34] D. Ousmane Samary and F. Vignes-Tourneret, “Just Renormalizable TGFT’s on  $U(1)^d$  with Gauge Invariance,” *Commun. Math. Phys.* DOI:10.1007/s00220-014-1930-3 (2014) [arXiv:1211.2618 [hep-th]].
- [35] S. Carrozza, D. Oriti and V. Rivasseau, “Renormalization of an  $SU(2)$  Tensorial Group Field Theory in Three Dimensions,” *Commun. Math. Phys.* DOI:10.1007/s00220-014-1928-x (2014) [arXiv:1303.6772 [hep-th]].
- [36] J. Ben Geloun, “Two and four-loop  $\beta$ -functions of rank 4 renormalizable tensor field theories,” *Class. Quant. Grav.* **29**, 235011 (2012) [arXiv:1205.5513 [hep-th]].
- [37] D. Ousmane Samary, “Beta functions of  $U(1)^d$  gauge invariant just renormalizable tensor models,” *Phys. Rev. D* **88**, no. 10, 105003 (2013) [arXiv:1303.7256 [hep-th]].
- [38] S. Carrozza, “Discrete Renormalization Group for  $SU(2)$  Tensorial Group Field Theory,” arXiv:1407.4615 [hep-th].
- [39] H. Grosse and R. Wulkenhaar, “Renormalization of  $\phi^4$  theory on noncommutative  $\mathbb{R}^4$  in the matrix base,” *Commun. Math. Phys.* **256**, 305 (2005) [hep-th/0401128].
- [40] H. Grosse and R. Wulkenhaar, “Renormalization of  $\phi^4$  theory on noncommutative  $\mathbb{R}^2$  in the matrix base,” *JHEP* **0312**, 019 (2003) [hep-th/0307017].

- [41] V. Rivasseau, F. Vignes-Tourneret and R. Wulkenhaar, “Renormalization of noncommutative  $\phi^4$ -theory by multi-scale analysis,” Commun. Math. Phys. **262**, 565 (2006) [hep-th/0501036].
- [42] H. Grosse and R. Wulkenhaar, “The beta function in duality covariant noncommutative  $\phi^4$  theory,” Eur. Phys. J. C **35**, 277 (2004) [hep-th/0402093].
- [43] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, “Vanishing of Beta Function of Non Commutative  $\Phi_4^4$  Theory to all orders,” Phys. Lett. B **649**, 95 (2007) [hep-th/0612251].
- [44] H. Grosse and R. Wulkenhaar, “Progress in solving a noncommutative quantum field theory in four dimensions,” arXiv:0909.1389 [hep-th].
- [45] H. Grosse and R. Wulkenhaar, “Self-dual noncommutative  $\phi^4$ -theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory,” Commun. Math. Phys. **329** p 1069-1130 arXiv:1205.0465 [math-ph].
- [46] H. Grosse and R. Wulkenhaar, “Solvable 4D noncommutative QFT: phase transitions and quest for reflection positivity,” arXiv:1406.7755 [hep-th].
- [47] L. Comtet, “Advanced Combinatorics” D. Reidel Publishing Company.
- [48] D. Ousmane. Samary “Closed equations of the two-point functions for tensorial group field theory,” Class. Quant. Grav. **31**, 185005 (2014) arXiv:1401.2096 [hep-th].
- [49] V. Rivasseau, *From Perturbative to Constructive Renormalization*. Princeton series in physics. Princeton Univ. Pr., 1991. 336 p.